

A NOTE ON THE ANNIHILATOR OF LOCAL COHOMOLOGY MODULES IN CHARACTERISTIC ZERO

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ABSTRACT. We give an alternative proof to the annihilator of local cohomology in characteristic zero which was proved by Lyubeznik.

1. INTRODUCTION

Let R be a commutative Noetherian ring and $I \subset R$ be an ideal. The i th local cohomology module of R with support in I is denoted by $H_I^i(R)$. In the present note, our main result (Theorem 2.3) provides a different point of view of $\text{Ann}_R H_I^i(R) = 0$ credited to Lyubeznik [3, Corollary 3.6], where R is a regular local ring containing a field of characteristic 0. Our way to prove the results is to use the so-called D -modules. This method has played a decisive role in many subsequent studies in the rings of characteristic zero.

Let R be a commutative algebra over a field k of characteristic zero. We denote by $\text{End}_k(R)$ the k -linear endomorphism ring. The ring of k -linear differential operators $D_{R|k} \subseteq \text{End}_k(R)$ generated by the k -linear derivations $R \rightarrow R$ and the multiplications by elements of R . By a $D_{R|k}$ -module we always mean a left $D_{R|k}$ -module. The injective ring homomorphism $R \rightarrow D_{R|k}$ that sends r to the map $R \rightarrow R$ which is the multiplication by r , gives $D_{R|k}$ a structure of R -algebra. Every $D_{R|k}$ -module M is automatically an R -module via this map. The natural action of $D_{R|k}$ on R makes R a $D_{R|k}$ -module. If $R = k[[x_1, \dots, x_n]]$ is a formal power series ring of n variables x_1, \dots, x_n over k , then $D_{R|k}$ is left and right Noetherian. Moreover, $D_{R|k}$ is a simple ring. Noteworthy, the local cohomology module $H_I^i(R)$, $i \in \mathbb{Z}$ is a finitely generated $D_{R|k}$ -module. For a more advanced exposition based on differential operators and undefined concepts the interested reader might consult [1]. For brevity we often write D_R for $D_{R|k}$ when there is no ambiguity about the field k .

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2. RESULTS

Lemma 2.1. *Let R be $k[[x_1, \dots, x_n]]$, a formal power series ring of n variables x_1, \dots, x_n over a field k of characteristic zero. Suppose that $H_I^i(R) \neq 0$, then $\text{Ann}_{D_R} H_I^i(R) = 0$.*

Proof. Consider the homomorphism

$$(2.1) \quad D_R \xrightarrow{f} \text{Hom}_k(H_I^i(R), H_I^i(R))$$

of D_R -modules defined by $f(P)(m) = Pm$, $P \in D_R$ and $m \in H_I^i(R)$ for all $i \in \mathbb{Z}$. The homomorphism f is injective, as D_R is a simple ring, i.e. $\text{Ann}_{D_R} H_I^i(R) = \ker f = 0$. \square

Lemma 2.2. *Let R be as in Lemma 2.1. Let M be both an R module and a D_R -module. Then $\text{Ann}_{D_R} M = 0$ implies $\text{Ann}_R M = 0$.*

Proof. Let $r \in \text{Ann}_R M$ be an arbitrary element. As the endomorphism $\varphi_r : R \rightarrow R$ with $\varphi_r(s) = rs$ for all $s \in R$ is an element of D_R so from $rsM = 0$ (for all $s \in R$) we have $\varphi_r(s)M = 0$. That is $\varphi_r(s) \in \text{Ann}_{D_R} M = 0$, i.e. $rs = \varphi_r(s) = 0$, for all $s \in R$. Hence, we have $r = 0$, as desired. \square

Theorem 2.3. *Let (R, \mathfrak{m}) be a regular local ring containing a field of characteristic zero. Suppose that $H_I^i(R) \neq 0$. Then $\text{Ann}_R H_I^i(R) = 0$.*

Proof. Suppose $k = R/\mathfrak{m}$, where \mathfrak{m} is the maximal ideal of R and $\text{char}(k) = 0$. By virtue of [2, Chapter IX, Appendice 2.] there exists a faithfully flat homomorphism from (R, \mathfrak{m}) to a regular local ring (S, \mathfrak{n}) such that S/\mathfrak{n} is the algebraic closure of k . As S is faithfully flat over R then the homomorphism is injective so S contains a field. Moreover, it is known that $\text{Ann}_R H_I^i(R) = (\text{Ann}_S H_{IS}^i(S)) \cap R$ and $H_I^i(R) \otimes_R S \cong H_{IS}^i(S) \neq 0$, because of faithfully flatness of S . Then, we may assume that k is algebraically closed. As \hat{R} is also faithfully flat R -module, we may assume that R is complete, so that $R = k[[x_1, \dots, x_n]]$ by the Cohen Structure Theorem, where $n = \dim R$. Since, $k[[x_1, \dots, x_n]]$ has a D_R -module structure so, we are done by Lemma 2.2 and Lemma 2.1. \square

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